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Bifurcation of Kovalevskaya Polynomial

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1. INTRODUCTION

The rotation of a rigid body about a fixed point in the Kovalevskaya case, where $A = B = 2C$, $y_0 = z_0 = 0$ (A, B, C are the principal moments of inertia; x_0, y_0, z_0 represent the center of mass), has been reduced to quadrature, and the system can be integrated to a Riemann θ-function of two variables (Kovalevskaya, 1889).

The qualitative investigation of the motion of Kovalevskaya tops has been undertaken by many authors, starting with Applort (1940) and continuing with Kozlov (1975, 1980).

Kolossoff (1903) transformed the Kovalevskaya problem into plane motion under a certain potential force. By using elliptic coordinates, Kolossoff proved the inverse problem, i.e., he converted the plane motion system into a Kovalevskaya system.

The qualitative investigation of the motion in the two-dimensional tori is given here in order to obtain the bifurcation and the phase portrait of the problem.

2. THE KOVALEVSKAYA EQUATIONS

The equations of motion of a rigid body rotating about a fixed point in the Kovalevskaya case can be written in the form (Golubef, 1953)

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$$
\begin{aligned}
2\frac{dx_1}{d\tau} &= (-rx_1 - \gamma_3)i \\
2\frac{dx_2}{dt} &= (rx_2 + \gamma_3)i \\
2\frac{dr}{dt} &= (\xi_2 - \xi_1 + x_1^2 - x_2^2)i \\
\frac{d\xi_1}{dt} &= -\, r\xi_1 i \\
\frac{d\xi_2}{dt} &= r\xi_2 i \\
\frac{d\gamma_3}{dt} &= (x_2\xi_1 - x_1\xi_2 + x_1x_2(x_1 - x_2))i
\end{aligned} \tag{1}
$$

where the first integrals of the problem are

$$
\begin{aligned}
\xi_1\xi_2 &= k^2 \\
\gamma_3^2 &= 1 - k^2 + x_2^2\xi_1 + x_1^2\xi_2 - x_1^2x_2^2 \\
r^2 &= 6\ell_1 + \xi_1 - \xi_2 - (x_1 + x_2)^2 \\
r\gamma_3 &= 2\ell - x_2\xi_1 - x_2 + x_1x_2(x_1 + x_2)
\end{aligned} \tag{2}
$$

By using Jacobi's last multiplier, the system (1) can be reduced to quadrature,

$$
\begin{aligned}
\frac{ds_1}{\sqrt{\varphi(s_1)}} + \frac{ds_2}{\sqrt{\varphi(s_2)}} &= 0 \\
\frac{s_1\, ds_1}{\sqrt{\varphi(s_1)}} + \frac{s_2\, ds_2}{\sqrt{\varphi(s_2)}} &= \frac{dt}{2}
\end{aligned} \tag{3}
$$

where the Kovalevskaya polynomial $\varphi(s)$ is defined as

$$
\varphi(s) = 8\{s[(s - 3\ell_1)^2 + 1 - k^2] - 2\ell^2\}\{(s - 3\ell_1 - k)(s - 3\ell_1 + k)\}
$$

The system (3) can be also written in the form

$$
\begin{aligned}
\frac{ds_1}{\sqrt{\varphi(s_1)}} &= \frac{\sqrt{2}}{s_1 - s_2}\, dt \\
\frac{ds_2}{\sqrt{\varphi(s_2)}} &= \frac{\sqrt{2}}{s_1 - s_2}\, dt
\end{aligned} \tag{4}
$$

Kolossoff (1903) introduced the potential

$$U = \frac{\rho^2 - kx + 1}{\rho}, \qquad \rho = (x^2 + y^2)^{1/2} \tag{5}$$

to reduce the system (1) to the plane motion system defined as

$$\begin{aligned} x'' &= \frac{\partial U}{\partial x} \\ y'' &= \frac{\partial U}{\partial y} \end{aligned} \tag{6}$$

($' \equiv d/d\tau$), where the relation between the old and the new times t and T is

$$d\tau = \frac{2(rx_1 + \gamma_3)(rx_2 + \gamma_3)}{(x_1 - x_2)^2}\, dt \tag{7}$$

and the constants of energy in the two systems must be equal, i.e., $3\ell_1 = h$.

By taking the elliptic coordinates λ, μ such that

$$x = \frac{\lambda\mu}{k} + k$$

$$y = \frac{1}{k}\,[(\lambda^2 - k^2)(k^2 - \mu^2)]^{1/2}$$

with Jacobian matrix M defined as

$$M = \begin{pmatrix} \partial x/\partial\lambda & \partial x/\partial\mu \\ \partial y/\partial\lambda & \partial y/\partial\mu \end{pmatrix}$$

one can give the momenta p_x, p_y as a function of p_λ, p_μ, such that

$$\begin{pmatrix} p_x \\ p_y \end{pmatrix} = M^{-1}\begin{pmatrix} p_\lambda \\ p_\mu \end{pmatrix}$$

and so we get

$$p_x = \frac{(\lambda^2 - k^2)\mu p_\lambda - (\mu^2 - k^2)\lambda p_\mu}{k(\lambda^2 - \mu^2)}$$

$$p_y = \frac{[(\lambda^2 - k^2)(k^2 - \mu^2)]^{1/2}(\lambda p_\lambda - \mu p_\mu)}{k(\lambda^2 - \mu^2)}$$

The system is a Liouville system, where the constants of integration $3\ell_1 = h$ and C are found from

$$\begin{aligned} &(\lambda^2 - k^2)p_\lambda^2 + 2\lambda^3 + 2(1 - k^2)\lambda - 2\lambda^2 h \\ &\quad = (\mu^2 - k^2)p_\mu^2 + 2\mu^3 + 2(1 - k^2)\mu - 2\mu^2 h = C \end{aligned} \tag{8}$$

and the momenta p_λ and p_μ are given by

$$p_\lambda = \left(\frac{\varphi(\lambda)}{\lambda^2 - k^2}\right)^{1/2}, \qquad p_\mu = \left(\frac{\varphi(\mu)}{\mu^2 - k^2}\right)^{1/2} \tag{9}$$

By using the Hamilton–Jacob equations, we obtain

$$\begin{aligned} \frac{d\lambda}{[\varphi(\lambda)(\lambda^2 - k^2)]^{1/2}} + \frac{d\mu}{[\varphi(\mu)(\mu^2 - k^2)]^{1/2}} &= 0 \\ \frac{\lambda d\lambda}{[\varphi(\lambda)(\lambda^2 - k^2)]^{1/2}} + \frac{\mu d\mu}{[\varphi(\mu)(\mu^2 - k^2)]^{1/2}} &= \frac{d\tau}{\sqrt{2}(\lambda + \mu)} \end{aligned} \tag{10}$$

By using the change of time

$$\frac{d\tau}{\sqrt{2}(\lambda + \mu)} = dt \tag{11}$$

the equations (10) become the Kovalevskaya equations (3), where the polynomial

$$P(u) = (u^2 - k^2)\varphi(u)$$

corresponds to the Kovalevskaya polynomial $\varphi(s)$.

Returning to the Kovalevskaya system (4), introducing the time τ defined in (11), we have

$$\begin{aligned} \frac{ds_1}{\sqrt{\varphi(s_1)}} &= \frac{d\tau}{s_1^2 - s_2^2} \\ \frac{ds_2}{\sqrt{\varphi(s_2)}} &= \frac{d\tau}{s_2^2 - s_1^2} \end{aligned} \tag{12}$$

If we put $s_i = s_i - 3\ell$, then the function $\varphi\ (s)$ takes the form

$$\varphi(s) = -\{(s + h)(s^2 + 1 - k^2) - 2\ell^2\}(s^2 - k^2)$$

and equations (12) become

$$\begin{aligned} \frac{ds_1}{\sqrt{\varphi(s_1)}} + \frac{ds_2}{\sqrt{\varphi(s_2)}} &= 0 \\ \frac{(s_1^2 + 1 - k^2)\, ds_1}{\sqrt{\varphi(s_1)}} + \frac{(s_2^2 + 1 - k^2)\, ds_2}{\sqrt{\varphi(s_2)}} &= d\tau \end{aligned}$$

By integrating, we have

$$\int_{s_{10}}^{s_1} \frac{ds_1}{\sqrt{\varphi(s_1)}} + \int_{s_{20}}^{s_2} \frac{ds_2}{\sqrt{\varphi(s_2)}} = \text{const}$$

$$\int_{s_{10}}^{s_1} \frac{(s_1^2 + 1 - k^2)\, ds_1}{\sqrt{\varphi(s_1)}} + \int_{s_{20}}^{s_2} \frac{(s_2^2 + 1 - k^2)\, ds_2}{\sqrt{\varphi(s_2)}} = \tau - \tau_0 \tag{13}$$

Now if we define a function V such that

$$V = \int_{s10}^{s_2} \left(\frac{s_1^3 + s_1(1 - k^2) + h(s_1^2 + 1 - k^2) - 2\ell^2}{k^2 - s_1^2}\right)^{1/2} ds_1 + \int_{s_{20}}^{s_2} \left(\frac{s_2^3 + s_2(1 - k^2) + h(s_2^2 + 1 - k^2) - 2\ell^2}{k^2 - s_2^2}\right)^{1/2} ds_2 \tag{14}$$

then it is easy to find

$$\frac{\partial V}{\partial \ell^2} = \text{const}$$

$$\frac{\partial V}{\partial h} = \tau - \tau_0$$

which are the same as equations (13). So if we write the function V in the form

$$V = \int p_1\, ds_1 + \int p_2\, ds_2 \tag{15}$$

then we have the following relations:

$$s_1^3 + s_1(1 - k^2) + h(s_1^2 + 1 - k^2) - 2\ell^2 = p_1^2(k^2 - s_1^2)$$
$$s_2^3 + s_2(1 - k^2) + h(s_2^2 + 1 - k^2) - 2\ell^2 = p_2^2(k^2 - s_2^2) \tag{16}$$

and the function V becomes the complete integral of the Hamilton–Jacob equations

$$H\left(\frac{\partial V}{\partial s_i}, s_i\right) = h \tag{17}$$

Equation (17) can be separated and the problem reduced to quadrature. Solving the inversion problem (17), we get the Kovalevskaya problem.

The above discussion allows us to study the bifurcation and the phase portrait of the two-dimensional invariant tori of Kovalevskaya's problem, where the Kolossoff variables are the same as the Euler–Poisson variables.

3. PHASE PORTRAIT OF THE SEPARATED FUNCTIONS

Consider the function

$$f = q^3 + q(1 - k^2) + h(q^2 + 1 - k^2) - p^2(k^2 - q^2) \tag{18}$$

We shall construct the lines of constant f on the plane (p, q), which is called the phase portrait of f. The phase portrait helps us to find the topological interpretation of the trajectory as follows: if the roots of the function are distinct for given initial values of (p, q), then $P(p_1, p_2)$ and $Q(q_1, q_2)$ will change in a periodic manner, but if the function has multiple roots, then we have an infinity of motion, which gives an asymptotic solution of the canonical equations. Thus the study of lines of constant f provides a complete picture of the bifurcations of Kovalevskaya polynomial.

To construct the lines of constant f, we first study the singular points of f. These points can be found from the equations

$$\frac{\partial f}{\partial p} = -p(k^2 - q^2) = 0 \tag{19}$$

$$\frac{\partial f}{\partial q} = 3q^2 + 2hq + 1 - k^2 + p^2 q = 0 \tag{20}$$

and hence we have the following: where $p = 0$ we get

$$3q^2 + 2hq + 1 - k^2 = 0 \tag{21}$$

and when $p = 0$, $q = k$, we get the two equations

$$p^2 = -h - k - \frac{1}{2k} \tag{22}$$

$$p^2 = -h + k + \frac{1}{2k} \tag{23}$$

$(k > 0)$. The positive regions of the functions

$$f_1 = h^2 + 3k^2 - 1$$

$$f_2 = -2hk - 2k^2 - 1$$

$$f_3 = -2hk + 2k^2 + 1$$

are shown in Fig. 1. It is clear that the curve $f_1 = 0$ is tangent to the branches of the curves f_2 and f_3 at the points $k = \pm 1/2$. This can be found from the consideration that the equation $f_1 = f_2$ gives the two roots (k_1, h_1), (k_2, h_2) such that

$$\operatorname{sign} k_1 \cdot \operatorname{sign} k_2 < 0$$

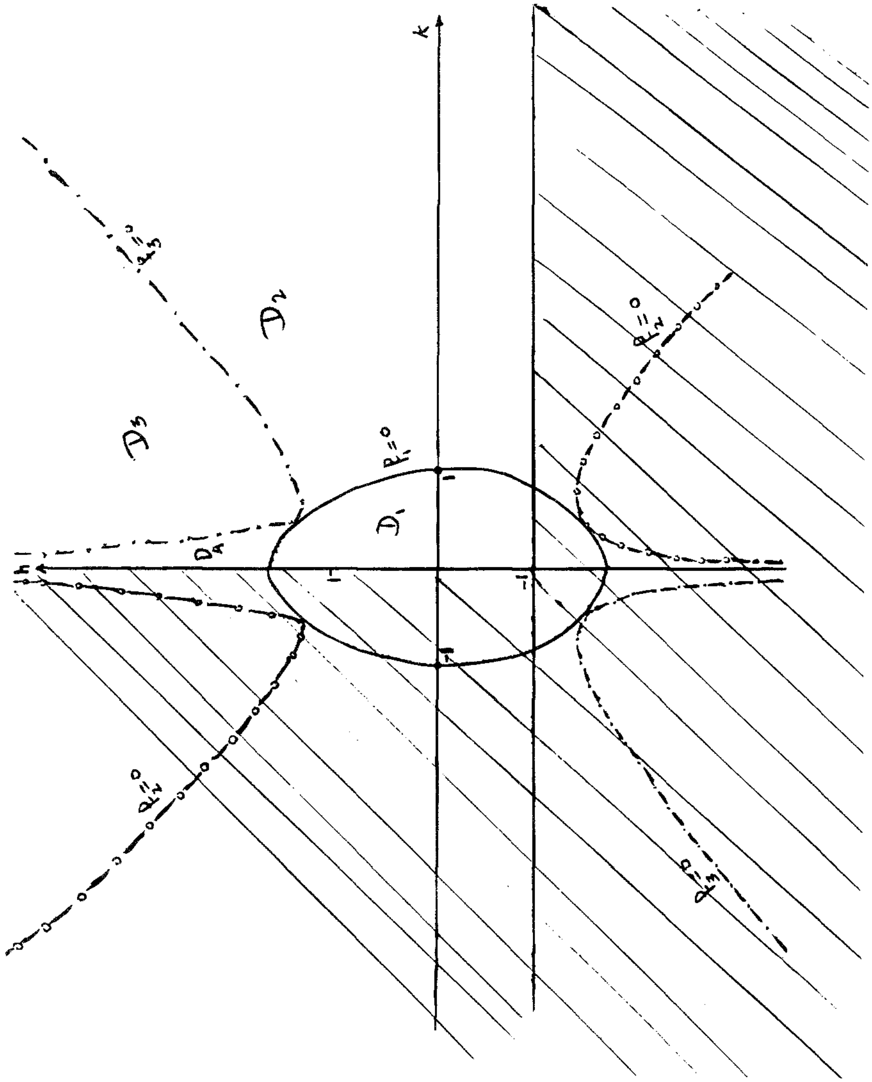

Fig. 1. The regions D_i of real motions on the (k, h) plane.

and from $f_2 = 0$ we have $h = -k - 1/2k$; substituting in the equation $f_1 = 0$, we get

$$4k^2 + \frac{1}{4k^2} - 2 = 0$$

In the same manner, we get the points between the curves $f_1 = 0$ and the branches of f_2.

We study the motion in domain D_i ($i = 1, 2, 3, 4$), where the real motion occurs in the regions $\{k > 0, h > -1\}$.

1. The First Region D_1: $f_1 < 0, f_2 < 0, f_3 > 0$

Equations (21) and (22) are not solved when $p = 0$ and $q = k$, and hence there are no singular points of f on the line $p = 0$ and $q = k$, while when $q = -k$, we have two singular points with p coordinates

$$p = \pm\left(\frac{2k^2 - 2hk + 1}{2k}\right)^{1/2}$$

To get these types of points, we put

$$q = -k + y, \qquad p = \pm\left(\frac{2k^2 - 2hk + 1}{2k}\right)^{1/2} + x$$

in the function f, neglecting terms of degree >2:

$$f = \frac{-k^2 - hk + 1}{k} y^2 \mp 4k\left(\frac{2k^2 - 2hk + 1}{2k}\right)^{1/2} xy + A_0$$

(A_0 is the value of f when $x = y = 0$).

The singular points are hyperbolic points, where

$$\begin{vmatrix} \partial^2 f/\partial x^2 & \partial^2 f/\partial x\, \partial y \\ \partial^2 f/\partial y\, \partial x & \partial^2 f/\partial y^2 \end{vmatrix} < 0 \qquad \text{when} \quad x = y = 0$$

The phase portrait of the function f in the domain D_1 is shown in Fig. 2. The lines $p = 0$, $q = +k$ are the lines of constant function f. The phase portrait is symmetric with respect to the line $p = 0$.

2. The Domain D_2: $f_1 > 0, f_2 < 0, f_3 > 0$

On the line $p = 0$, there are two singular points:

$$\left(\frac{-h + (h^2 + 3k^2 - 3)^{1/2}}{3}, 0\right), \qquad \left(\frac{-h - (h^2 + 3k^2 - 3)^{1/2}}{3}, 0\right)$$

The singular points become $(\pm[(k^2 - 1)^{1/2}/3], 0)$ with the conditions $h = 0$, $k > 1$. The function f takes the form

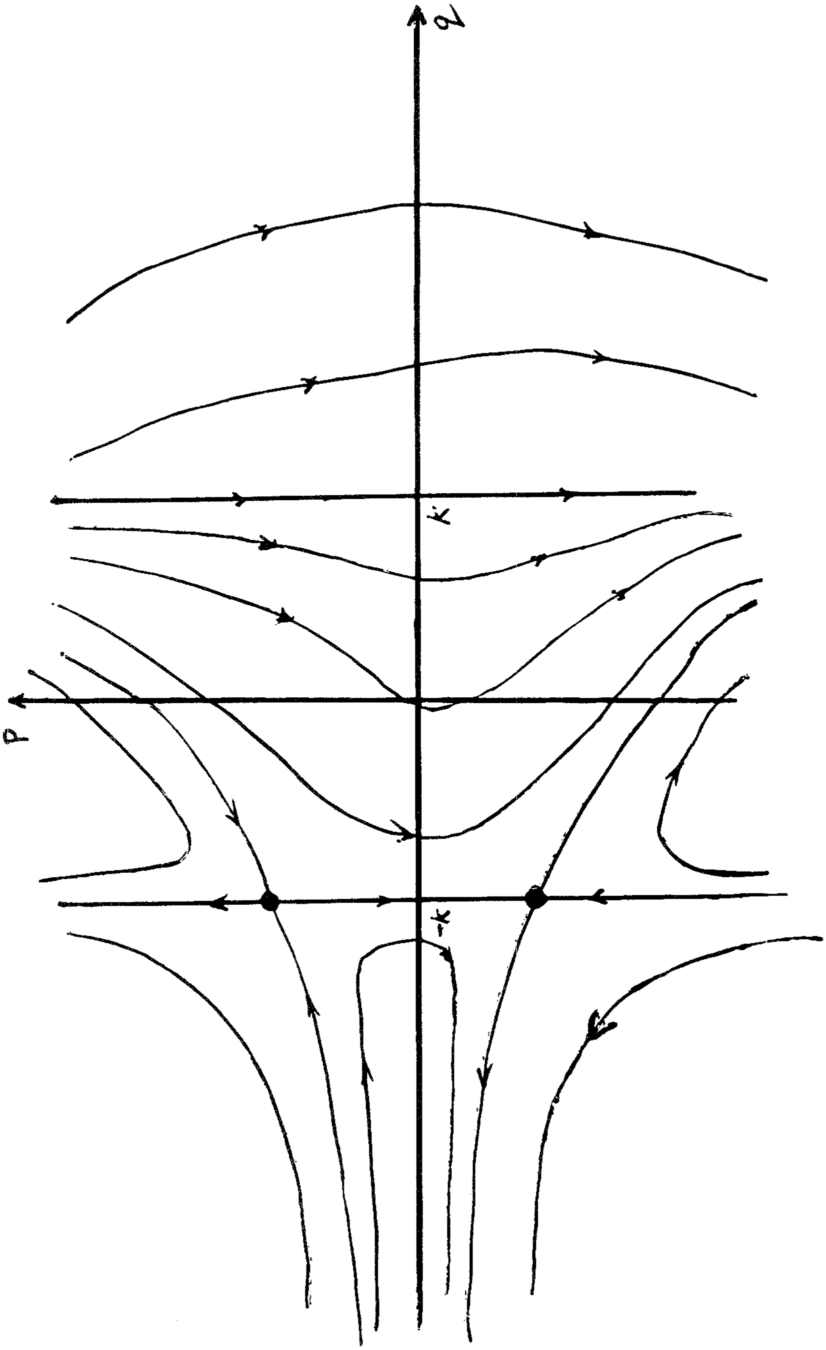

Fig. 2. The phase portrait of f in the region D_1.

$$f = \pm[3(k^2 - 1)]^{1/2} y^2 - \frac{2k^2 + 1}{3} x^2$$

where

$$p = x, \qquad q = \pm\left(\frac{k^2 - 1}{3}\right)^{1/2} + y$$

There are two hyperbolic points on the line $q = -k$, while on the line $q = k$ there are no singular points.

Figure 3 shows the phase portrait on the domain D_2. It can be noted that the points 1, 2, and 4 are hyperbolic and point 3 is elliptic.

3. In the Domain D_3: $f_1 > 0, f_2 < 0, f_3 < 0$

At $p = 0$ we have the two points with q coordinates

$$q_{1,2} = \frac{-h \pm (h^2 - 3k^2 + 3)^{1/2}}{3}$$

We take the case $k = 1, h >> 0$ and the points become $(-2h/3, 0)$ and $(0, 0)$.

The function f takes the form

$$f = -hy^2 - x^2\left(k^2 - \frac{4h^2}{9}\right)$$

The phase portrait is shown in Fig. (4), where the points 5 and 6 are hyperbolic points

4. In the Domain D_4: $f_1 > 0, f_2 < 0, f_3 > 0$

Finally in the domain D_4, we have the points

$$p = 0, \qquad q_{1,2} = \frac{-h + (h^2 - 3k^2)^{1/2}}{3}$$

and we get the phase portrait of f which shown in Fig. 5, where the points 7 and 8 are elliptic and hyperbolic, respectively.

We conclude with some results related to the behavior of the motion:

1. There are no multiple roots of the function f, and consecutively p, q are changing periodically.

2. The elliptic points in the figures are stable in the Lyapunov sense, because a small disturbance will result in a closed trajectory that surrounds it and along which the state of the system remains close to these points.

3. The hyperbolic points are unstable because any small disturbance will result in a trajectory on which the state of the system deviates more and more from these points as t goes to infinity.

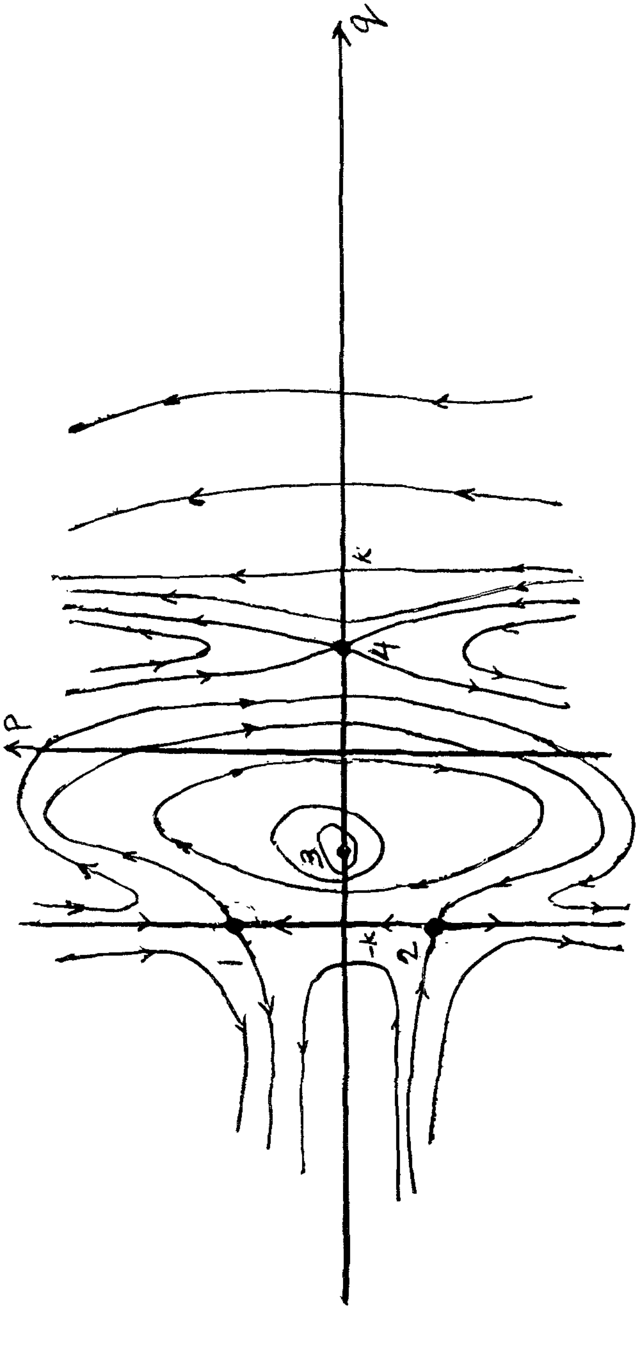

Fig. 3. The phase portrait of f in the region D_2.

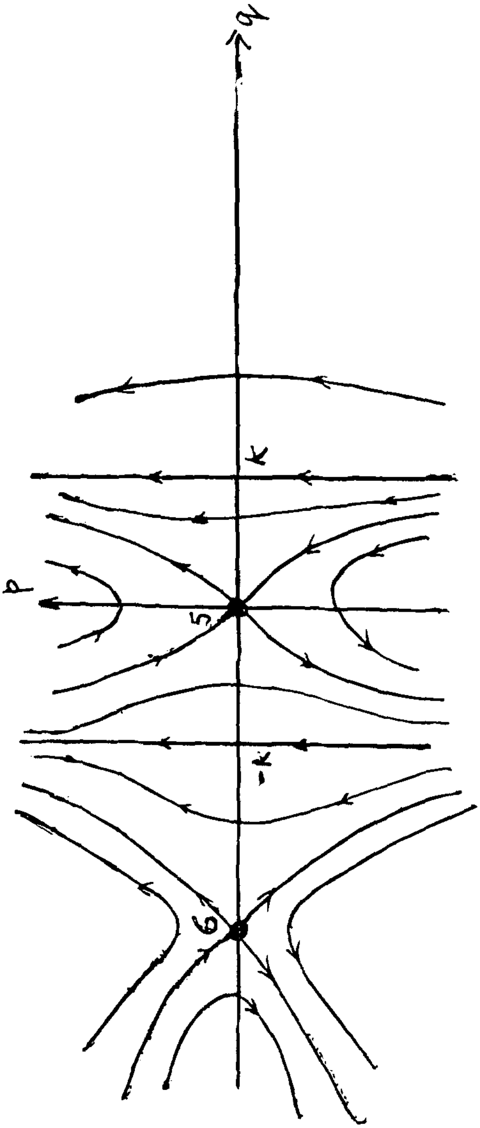

Fig. 4. The phase portrait of f in the region D_3.

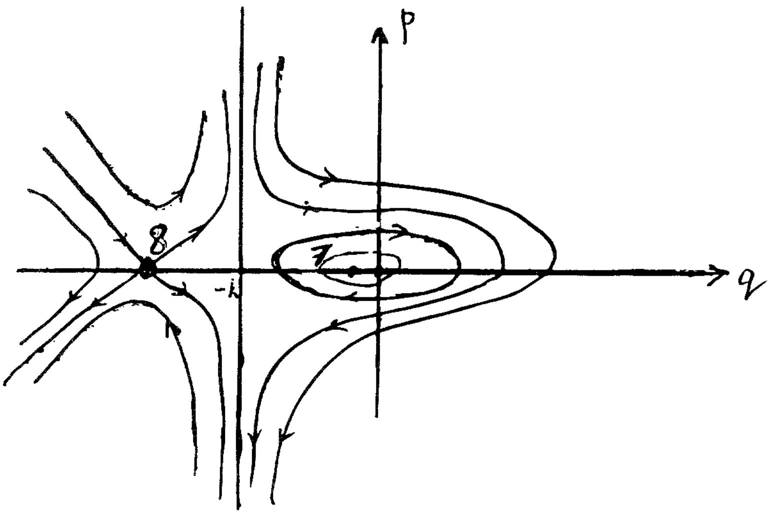

Fig. 5. The phase portrait of f in the region D_4.

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